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1981 J. Phys. A: Math. Gen. 14 2281

(http://iopscience.iop.org/0305-4470/14/9/023)

J. Phys. A: Math. Gen. 14 (1981) 2281–2303. Printed in Great Britain

Nonlinear scattering processes in the presence of a quantised radiation field: II. Relativistic treatment

J Bergou and S Varró
Central Research Institute for Physics, H-1525 Budapest 114, POB 49, Hungary

Received 22 October 1980

Abstract. The Dirac equation of a relativistic free charged particle interacting with one quantised mode of the electromagnetic field is solved exactly. Stationary eigenstates and the corresponding eigenvalues (the spectrum) are obtained in closed analytical form for linear and circular polarisation. The states are parametrised by two quantum numbers, one of which, corresponding to the four-momentum, is continuous and the other, roughly corresponding to the photon content, discrete. Based on these results, the cross section of nonlinear Compton scattering is calculated. It is shown explicitly that the results reduce to the semiclassical one in the limit of high intensity and small depletion, while in the large (complete) depletion limit they contain a depletion factor which ensures convergence of the highly nonlinear processes.

1. Introduction

This paper forms the second part of a series devoted to the theoretical description of the interaction between a free electron and an arbitrarily intense radiation field mode. In the first part (Bergou and Varró 1981, henceforth referred to as I) we dealt with a model where the electron was described by non-relativistic quantum mechanics and the vector potential specifying the mode of the radiation field was considered as a quantised field quantity. The present work is intended to generalise the results of I to the relativistic case, insofar as the electron will be described by relativistic quantum mechanics (Dirac equation), and the dipole approximation for the field will never be introduced. In this sense our model occupies an intermediate place between semiclassical electrodynamics and QED. Namely, we consider the radiation field quantum mechanically and not as a classical quantity, but from the beginning we use the one-mode approximation, and second quantisation is not introduced for the electron field. This is not merely an academic exercise; the range of intensities in which our results give a reasonably good approximation is just the one which is of practical importance in the description of the laser beam–matter interaction, and we think it useful that accurate theoretical predictions be accessible without having to carry out full QED analysis. The only feature missing from our calculations is pair creation; with optical photons this can almost always be neglected. Generalisation of the model to several independent modes is straightforward; however, neglecting the mode–mode coupling still remains questionable. Nevertheless, the model fills in a gap between QED and semiclassical electrodynamics, since our calculations may serve as a starting point for intense-field QED

0305-4470/81/092281+23$01.50

calculations and, in addition, they provide the foundation of semiclassical approximations, since in the limit of large initial photon number and small depletion, they reduce to the results of the semiclassical theory.

The starting point of our method is the exact solution to the Dirac equation of the system 'electron + quantised electromagnetic mode'. The basic idea is to introduce a properly chosen unitary transformation—a so-called displacement transformation—in order to eliminate the interaction term and diagonalise the system of equations. A similar transformation was introduced by Bloch and Nordsieck (1937) when they solved the problem of elimination of infrared divergences from QED. In § 2 we give the solution of the eigenvalue problem and stationary states of the system 'electron + linearly polarised mode'. The first solution to this problem was given by Berson (1969), using a special representation for the absorption and emission operators. The main advantage of our method is that the results are derived by purely algebraic methods in a representation-independent manner. In § 3 we repeat the procedure for a circularly polarised mode. The solution in this case was made possible by the use of the projection technique (Neville and Rohrlich 1971a, b, Becker and Mitter 1974). Essentially similar problems were treated in the mid-sixties (Fried and Eberly 1964, Eberly and Reiss 1966, Reiss and Eberly 1966, Eberly 1969); however, in these papers attention was focused on the direct calculation of the Green function rather than on the determination of stationary states of the system. Furthermore, depletion of the mode (i.e. change of the photon number) was neglected in the otherwise exact summation of the diagrams for the Green function, and thus the method was in fact equivalent to the semiclassical calculations. We note that our results in the aforementioned semiclassical limit reduce to the Volkov solution (Volkov 1935), and therefore give the generalisation of the semiclassical wavefunction. In § 4 we discuss statistical properties of the electron and photon subsystems. Special attention is paid to the role of the spin–field interaction. Section 5 deals with an application of the results obtained in §§ 2 and 3. In a previous publication (Bergou and Varró 1980) we outlined a semiclassical method to deal with intense-field problems. The method was based on the use of the exact Volkov solution of the electron–external field problem as a basis set for perturbational treatment of scattering processes. Here we generalise the method by using the exact solutions of §§ 2 and 3, and calculate the cross sections of Compton scattering and higher harmonics generation. The first such calculation based on the use of the Volkov solution was performed by Alperin (1944), using the method of transition currents. Later, with the development of high-power lasers, the problem was reinvestigated (Goldman 1964, Brown and Kibble 1964, Nikishov and Ritus 1964), also on semiclassical grounds. Our results reproduce the semiclassical ones in the intense-field limit and give an immediate generalisation since they contain depletion terms. Section 6 summarises the main results of the paper and discusses their physical implications as well as their connection with previous works.

2. A simple algebraic solution for the eigenvalue equation of the system 'electron + linearly polarised mode'

The Hamiltonian H of the system 'electron + one mode' consists of the energy operators of the subsystems and their interaction,

$$H = c\boldsymbol{\alpha}\hat{\boldsymbol{p}} + \beta mc^2 + \hbar\omega(a^+a + \tfrac{1}{2}) - e\boldsymbol{\alpha}\boldsymbol{A}(\boldsymbol{r}). \quad (2.1)$$

Here $\boldsymbol{A}(\boldsymbol{r})$ is the vector potential in the Schrödinger picture characterising the transverse photon with polarisation vector $\boldsymbol{\varepsilon}$ and wavevector $\boldsymbol{k}$:

$$\boldsymbol{A}(\boldsymbol{r}) = c\left(\frac{2\pi\hbar}{\omega V}\right)^{1/2} \boldsymbol{\varepsilon}(a\,\mathrm{e}^{\mathrm{i}\boldsymbol{kr}} + a^{+}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{kr}}), \qquad \boldsymbol{k\varepsilon} = 0, \qquad \omega = c|\boldsymbol{k}|. \tag{2.2}$$

a and a^{+} are absorption and emission operators of the given mode satisfying the commutation relation

$$aa^{+} - a^{+}a = 1 \tag{2.3}$$

and V is the quantisation volume. It can easily be shown that the operator of the total momentum of the system commutes with the Hamiltonian. As a consequence, there is a system of simultaneous eigenfunctions of the energy and momentum. The present section is devoted to the determination of these simultaneous eigenfunctions, i.e. we look for the solution $\psi_{E,\boldsymbol{P}}$ of the equations

$$H\psi_{E,\boldsymbol{P}} = E\psi_{E,\boldsymbol{P}} \tag{2.4}$$

and

$$[\hat{\boldsymbol{p}} + \hbar\boldsymbol{k}(a^{+}a + \tfrac{1}{2})]\psi_{E,\boldsymbol{P}} = \boldsymbol{P}\psi_{E,\boldsymbol{P}}. \tag{2.5}$$

If we express $\hat{\boldsymbol{p}}\psi_{E,\boldsymbol{P}}$ from (2.5) and—taking into account the explicit form of H as given by (2.1)—substitute it into (2.4), we obtain

$$\{c\boldsymbol{\alpha}[\boldsymbol{P} - \hbar\boldsymbol{k}(a^{+}a + \tfrac{1}{2})] + \beta mc^{2} + \hbar\omega(a^{+}a + \tfrac{1}{2}) - e\boldsymbol{\alpha\varepsilon}c(2\pi\hbar/\omega V)^{1/2}(a\,\mathrm{e}^{\mathrm{i}\boldsymbol{kr}} + a^{+}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{kr}})\}\psi_{E,\boldsymbol{P}} = E\psi_{E,\boldsymbol{P}}. \tag{2.6}$$

With the help of the unitary transformation $\exp[-\mathrm{i}\boldsymbol{kr}(a^{+}a + \frac{1}{2})]$, one can easily eliminate the dependence of the vector potential on space coordinates in (2.6), yielding

$$\{c\boldsymbol{\alpha}[\boldsymbol{P} - \hbar\boldsymbol{k}(a^{+}a + \tfrac{1}{2})] + \beta mc^{2} + \hbar\omega(a^{+}a + \tfrac{1}{2}) - e\boldsymbol{\alpha\varepsilon}c(2\pi\hbar/\omega V)^{1/2}(a + a^{+})\}\Phi_{E,\boldsymbol{P}} = E\Phi_{E,\boldsymbol{P}} \tag{2.7}$$

where

$$\Phi_{E,\boldsymbol{P}} = \exp[\mathrm{i}\boldsymbol{kr}(a^{+}a + \tfrac{1}{2})]\psi_{E,\boldsymbol{P}}. \tag{2.8}$$

By using the standard representation for $\boldsymbol{\alpha}$ and β (see equation (A1.3)) in equation (2.7), and introducing upper and lower spinor components—$\varphi(\boldsymbol{r})$ and $\chi(\boldsymbol{r})$ respectively—of $\Phi_{E,\boldsymbol{P}}$, we obtain the following coupled system of algebraic equations:

$$\{\boldsymbol{\sigma}[\boldsymbol{K} - \boldsymbol{k}(a^{+}a + \tfrac{1}{2})] - \boldsymbol{\sigma\varepsilon}g(a + a^{+})\}\chi(\boldsymbol{r}) + [\kappa - K_0 + k_0(a^{+}a + \tfrac{1}{2})]\varphi(\boldsymbol{r}) = 0, \tag{2.9a}$$

$$\{\boldsymbol{\sigma}[\boldsymbol{K} - \boldsymbol{k}(a^{+}a + \tfrac{1}{2})] - \boldsymbol{\sigma\varepsilon}g(a + a^{+})\}\varphi(\boldsymbol{r}) - [\kappa + K_0 - k_0(a^{+}a + \tfrac{1}{2})]\chi(\boldsymbol{r}) = 0, \tag{2.9b}$$

where we have employed the notations

$$\boldsymbol{K} = \boldsymbol{P}/\hbar, \qquad K_0 = E/\hbar c, \qquad k_0 = \omega/c \tag{2.10a}$$

and

$$\kappa = \frac{mc}{\hbar}, \qquad g = \frac{e}{\hbar}\left(\frac{2\pi\hbar}{\omega V}\right)^{1/2} = \frac{e}{(\hbar c)^{1/2}}\left(\frac{\lambda}{V}\right)^{1/2}, \qquad \lambda = \frac{2\pi c}{\omega}. \tag{2.10b}$$

At first sight, the system of equations (2.9a) and (2.9b) might seem rather complicated, since each component of the functions $\varphi(\boldsymbol{r})$ and $\chi(\boldsymbol{r})$ contains boson operators. If we choose the Majorana representation for the matrices $\boldsymbol{\alpha}$ and β, and choose a coordinate system where the polarisation and the wavevector of the electromagnetic mode coincide with the x and y axis, respectively, then instead of equations (2.9a) and (2.9b), we obtain a system of equations of considerably simpler structure. We also note at this point that, quite similarly to this case, the Dirac equation of the corresponding semiclassical problem is easily solvable without invoking the second-order Dirac equation (Bergou and Varró 1980).

Equation (2.7) reads in the Majorana representation (see (A1.6a) and (A1.6b))

$$\{\alpha_x[-K_x+g(a+a^+)]+\beta[K_y-k_0(a^+a+\tfrac{1}{2})]-\alpha_z K_z+\alpha_y\kappa+k_0(a^+a+\tfrac{1}{2})\}\Phi'_{E,\boldsymbol{P}}$$
$$=K_0\Phi'_{E,\boldsymbol{P}}, \qquad k_y\equiv k_0. \tag{2.11}$$

The upper and lower components $\varphi'(\boldsymbol{r})$ and $\chi'(\boldsymbol{r})$ of the transformed state $\Phi'_{E,\boldsymbol{P}}$ now satisfy the equations

$$\{[-K_x+g(a+a^+)]\sigma_x-K_z\sigma_z+\kappa\sigma_y\}\chi'(\boldsymbol{r})-(K_0-K_y)\varphi'(\boldsymbol{r})=0, \tag{2.12a}$$

$$\{[-K_x+g(a+a^+)]\sigma_x-K_z\sigma_z+\kappa\sigma_y\}\varphi'(\boldsymbol{r})-[K_0+K_y-2k_0(a^+a+\tfrac{1}{2})]\chi'(\boldsymbol{r})=0. \tag{2.12b}$$

We can simply express $\varphi'(\boldsymbol{r})$ from (2.12a) as

$$\varphi'(\boldsymbol{r})=(K_0-K_y)^{-1}\{[-K_x+g(a+a_+)]\sigma_x-K_z\sigma_z+\kappa\sigma_y\}\chi'(\boldsymbol{r}). \tag{2.12c}$$

Substitution of this expression for $\varphi'(\boldsymbol{r})$ into (2.12b) yields

$$\{K_x^2+K_y^2+K_z^2+\kappa^2-K_0^2-2gK_x(a+a^+)+g^2(a^2+a^{+2})$$
$$+2[g^2+(K_0-K_y)k_0](a^+a+\tfrac{1}{2})\}\chi'(\boldsymbol{r})=0. \tag{2.13}$$

Equation (2.13) serves as a starting point to the theory to be described in the following. We shall solve it by diagonalising the LHS in the basis of photon-number eigenstates.

In the first step, we eliminate the quadratic expression $g^2(a^2+a^{+2})$ with the help of the Bogolyubov transformation (Tanabe 1973)

$$C_\Theta\equiv\exp[-\tfrac{1}{2}\Theta(a^{+2}-a^2)] \tag{2.14}$$

where Θ is a real parameter to be determined later. The effect of C_Θ on the boson operators a and a^+ is the following:

$$C_\Theta^{-1}aC_\Theta=a\cosh\Theta-a^+\sinh\Theta, \qquad C_\Theta^{-1}a^+C_\Theta=a^+\cosh\Theta-a\sinh\Theta. \tag{2.14a}$$

Applying these transformation rules, after simple algebraic operations we obtain from (2.13) the transformed equation

$$\{K_x^2+K_y^2+K_z^2+\kappa^2-K_0^2-2gK_x(a+a^+)\,\mathrm{e}^{-\Theta}+g^2[(a^2+a^{+2})\cosh 2\Theta$$
$$-2(a^+a+\tfrac{1}{2})\sinh 2\Theta]+[g^2+(K_0-K_y)k_0][2(a^+a+\tfrac{1}{2})\cosh 2\Theta$$
$$-(a^2+a^{+2})\sinh 2\Theta]\}C_\Theta^{-1}\chi'(\boldsymbol{r})=0. \tag{2.15}$$

From here one can immediately see that if Θ is taken to satisfy the relationship

$$\tanh 2\Theta=g^2/[g^2+(K_0-K_y)k_0] \tag{2.15a}$$

then the terms containing (a^2+a^{+2}) in (2.15) cancel and we are left with the equation

$$\{K_x^2+K_y^2+K_z^2+\kappa^2-K_0^2-2gK_x(a+a^+)\,\mathrm{e}^{-\Theta}$$
$$+2[g^2+k_0(K_0-K_y)](a^+a+\tfrac{1}{2})\operatorname{sech}2\Theta\}C_\Theta^{-1}\chi'(\boldsymbol{r})=0. \tag{2.16}$$

Elimination of the terms containing $(a^+ + a)$ can be performed with the help of the well known displacement operator D_τ

$$D_\tau \equiv \exp[\tau(a^+ - a)] \tag{2.17}$$

having the property (Glauber 1963)

$$D_\tau^{-1} a D_\tau = a + \tau, \qquad D_\tau^{-1} a^+ D_\tau = a^+ + \tau. \tag{2.17a}$$

Here τ is a real parameter. If τ is chosen to satisfy the relationship

$$gK_x \, e^{-\Theta} = \tau[g^2 + k_0(K_0 - K_y)] \operatorname{sech} 2\Theta \tag{2.18}$$

then, applying D_τ to (2.16), we obtain

$$\{K_x^2 + K_y^2 + K_z^2 + \kappa^2 - K_0^2 + 2[g^2 + k_0(K_0 - K_y)] \times (a^+ a + \tfrac{1}{2} + \tau^2) \operatorname{sech} 2\Theta\} D_\tau^{-1} C_\Theta^{-1} \chi'(\boldsymbol{r}) = 0. \tag{2.19}$$

From here it transpires that the solution of (2.13) is of the form

$$\chi'(\boldsymbol{r}) = C_\Theta D_\tau |n\rangle \chi_0'(\boldsymbol{r}), \qquad n = 0, 1, 2, \ldots, \tag{2.20}$$

where $|n\rangle$ is a number state of the quantised mode and $\chi_0'(\boldsymbol{r})$ is a bispinor independent of the photon state.

The parameters Θ and τ of the unitary operators C_Θ and D_τ defined in (2.14) and (2.17) are determined by the relationships (2.15*a*) and (2.18). The four-momentum $K \equiv (K_0, \boldsymbol{K})$ satisfies the equation

$$K^2 = K_0^2 - \boldsymbol{K}^2 = \kappa^2 + 2(K_0 - K_y) k_0 g_K(n), \qquad n = 0, 1, 2, \ldots, \tag{2.21}$$

where

$$g_K(n) \equiv [1 + g^2/(K_0 - K_y)k_0](n + \tfrac{1}{2} - \tau^2) \operatorname{sech} 2\Theta. \tag{2.21a}$$

Determination of K_0 directly from (2.21) would still be rather difficult since $g_K(n)$ contains K_0 also through the parameter Θ. This difficulty, however, can easily be overcome if we introduce the four-vector Q as

$$Q \equiv (K_0 - k_0 g_K(n), K_x, K_y - k_0 g_K(n), K_z) = K - g_K(n) k. \tag{2.21b}$$

Since $Q_0 - Q_y = K_0 - K_y$ and $Q_x = K_x$, we have $g_K(n) = g_Q(n)$ from equations (2.21*b*), (2.19) and (2.21*a*). By using the definition (2.21*b*), from (2.21) one can easily see that Q is already on the free mass-shell,

$$Q^2 = \kappa^2, \tag{2.22}$$

and the four-momentum K of the system can be derived directly from

$$K = Q + k g_Q(n). \tag{2.23}$$

The solution of the eigenvalue equation (2.4) reads now, in the Majorana representation,

$$\psi'_{E,\boldsymbol{P}} \equiv \psi'_Q = \exp[-\mathrm{i}\boldsymbol{k}\boldsymbol{r}(a^+ a + \tfrac{1}{2})] \times \begin{bmatrix} (Q_0 - Q_y)^{-1}[-Q_x\sigma_x - Q_z\sigma_z + \kappa\sigma_y + g(a + a^+)\sigma_x]\chi_0'(\boldsymbol{r}) \\ \chi_0'(\boldsymbol{r}) \end{bmatrix} C_\Theta D_\tau |n\rangle. \tag{2.24a}$$

Quite similarly to $\psi_{E,\boldsymbol{P}}$, $\psi'_{E,\boldsymbol{P}}$ is also an eigenstate of the total momentum of the system;

therefore it satisfies equation (2.5). Substitution into (2.5) of $\psi'_{E,P}$ as given by (2.24a) leads to the equation

$$\hat{p}\chi'_0(\boldsymbol{r}) = \boldsymbol{P}\chi'_0(\boldsymbol{r}), \tag{2.24b}$$

the solution of which is

$$\chi'_0(\boldsymbol{r}) = \chi'_0 \exp[(\mathrm{i}/\hbar)\boldsymbol{Pr}]. \tag{2.24c}$$

Here χ'_0 is an arbitrary bispinor.

In order to determine the form of the state (2.24a) in standard representation, we notice that the bispinor

$$u'_Q = \begin{bmatrix} (Q_0 - Q_y)^{-1}(-Q_x\sigma_x - Q_z\sigma_z + \kappa\sigma_y)\chi'_0 \\ \chi'_0 \end{bmatrix} \tag{2.25a}$$

satisfies the transformed eigenvalue equation

$$(-\alpha_x Q_x + \beta Q_y - \alpha_z Q_z + \alpha_y\kappa)u'_Q = Q_0 u'_Q \tag{2.25b}$$

which is the eigenvalue equation for a free particle. Taking into account the definition (2.15a) of u'_Q, we may now write the bispinor amplitude of the solution (2.24a) as

$$\begin{bmatrix} (Q_0 - Q_y)^{-1}[-Q_x\sigma_x - Q_z\sigma_z + \kappa\sigma_y + g(a + a^+)\sigma_x]\chi'_0 \\ \chi'_0 \end{bmatrix}$$
$$= [1 + (1+\beta)\alpha_x g(a + a^+)/2(Q_0 - Q_y)]u'_Q. \tag{2.25c}$$

This becomes in standard representation

$$[1 - (1+\alpha_y)\alpha_x g(a + a^+)/2(Q_0 - Q_y)]u_Q. \tag{2.25d}$$

u_Q satisfies the eigenvalue equation of a free particle in standard representation:

$$(\boldsymbol{\alpha Q} + \beta\kappa)u_Q = Q_0 u_Q. \tag{2.26}$$

We also note that (2.25d) is a special case of the bispinor amplitude

$$[1 + g(\not{K}\not{\varepsilon}/2Qk)(a + a^+)]u_Q \tag{2.27}$$

written in a covariant manner (for the notation see appendix 1). The final form of $\psi_{E,P}$ can be given as

$$\psi_{E,P} = \psi_Q = \exp[-\mathrm{i}\boldsymbol{kr}(a^+a + \tfrac{1}{2})]\left(1 + g\frac{\not{K}\not{\varepsilon}}{2Qk}(a + a^+)\right)u_Q \exp\{\mathrm{i}[\boldsymbol{Q} + \boldsymbol{k}g_Q(n)]\boldsymbol{r}\}C_{\Theta}D_\tau|n\rangle. \tag{2.28}$$

The physical meaning of the quantum numbers Q and n which appear in the solution will be discussed in § 4. Roughly speaking, Q can be identified with the expectation value of the four-momentum of the electron, whereas n can be identified with the expectation value of the number of photons in the mode.

3. A simple algebraic solution for the eigenvalue equation of the system 'electron + circularly polarised mode'

The Hamiltonian form of the equation of motion of the system 'electron + quantised mode' is

$$\mathrm{i}\hbar\, \partial\Psi/\partial t = [c\boldsymbol{\alpha}\hat{\boldsymbol{p}} + \beta mc^2 + \hbar\omega(a^+a + \tfrac{1}{2}) - e\boldsymbol{\alpha A}(\boldsymbol{r})]\Psi \tag{3.1}$$

where now

$$\boldsymbol{A}(\boldsymbol{r}) = c(2\pi\hbar/\omega V)^{1/2}(\boldsymbol{\varepsilon} a\, \mathrm{e}^{\mathrm{i}\boldsymbol{kr}} + \boldsymbol{\varepsilon}^* a^+\, \mathrm{e}^{-\mathrm{i}\boldsymbol{kr}}), \qquad \boldsymbol{k\varepsilon} = 0. \tag{3.1a}$$

In the case of circular polarisation, $\boldsymbol{\varepsilon}$ is a complex unit vector (* denotes complex conjugation), which is defined for the right circular polarisation (+) and left circular polarisation (−) as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_+ = 2^{-1/2}(\boldsymbol{\varepsilon}_1 + \mathrm{i}\boldsymbol{\varepsilon}_2), \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_- = 2^{-1/2}(\boldsymbol{\varepsilon}_1 - \mathrm{i}\boldsymbol{\varepsilon}_2), \qquad \boldsymbol{\varepsilon}_1 \perp \boldsymbol{\varepsilon}_2, \tag{3.1b}$$

$$\boldsymbol{\varepsilon}^2 = \boldsymbol{\varepsilon}^{*2} = 0, \qquad \boldsymbol{\varepsilon\varepsilon}^* = 1. \tag{3.1c}$$

In the previous section on the example of the interaction with linearly polarised photons, we demonstrated that the stationary solutions of the equation of motion of the type (3.1) can be obtained by a relatively simple method. The two main steps were the application of the Bogolyubov transformation C_Θ and the application of the displacement transformation D_τ. In the case of circularly polarised photons we shall use a similar elimination technique.

For the sake of brevity and clarity we first bring equation (3.1) to covariant form. Let us introduce the state ψ with the definition

$$\Psi = \exp[-\mathrm{i}\omega(a^+a + \tfrac{1}{2})t]\psi. \tag{3.2}$$

Substituting this into (3.1), we obtain the equation of motion for ψ as

$$[\mathrm{i}\partial\!\!\!/ - g(\varepsilon\!\!\!/ a\, \mathrm{e}^{-\mathrm{i}kx} + \varepsilon\!\!\!/^* a^+\, \mathrm{e}^{\mathrm{i}kx}) - \kappa]\psi = 0. \tag{3.3}$$

For the notation see appendix 1. Here g is the coupling constant defined by equation (2.10b) with wavenumber dimension, κ is the Compton wavenumber and we have introduced the polarisation vector $\varepsilon \equiv \{\varepsilon^\mu\} = (0, \boldsymbol{\varepsilon})$. As a first step to the solution of equation (3.3), we eliminate the space- and time-dependent phasefactors $\mathrm{e}^{\pm\mathrm{i}kx}$ of the vector potential with the help of the unitary transformation

$$\psi = \exp[\mathrm{i}kx(a^+a + \tfrac{1}{2})]\Phi. \tag{3.4}$$

Then

$$[\mathrm{i}\partial\!\!\!/ - k\!\!\!/(a^+a + \tfrac{1}{2}) - g(\varepsilon\!\!\!/ a + \varepsilon\!\!\!/^* a^+) - \kappa]\Phi = 0. \tag{3.5}$$

We look for the solution of this equation in the form of a plane wave:

$$\Phi = \mathrm{e}^{-\mathrm{i}px}\Phi_p \tag{3.6}$$

and for Φ_p we obtain

$$[p\!\!\!/ - k\!\!\!/(a^+a + \tfrac{1}{2}) - g(\varepsilon\!\!\!/ a + \varepsilon\!\!\!/^* a^+) - \kappa]\Phi_p = 0. \tag{3.7}$$

Here p is a four-vector to be determined later.

Quite similarly to equation (2.27) of the previous section, we look for the solution of (3.7) with the ansatz

$$\Phi_p = [1 + (gk\!\!\!//2pk)(\varepsilon\!\!\!/ a + \varepsilon\!\!\!/^* a^+)]\chi_p. \tag{3.8}$$

Multiplying equation (3.7) by the expression $[1 - (gk\!\!\!//2pk)(\varepsilon\!\!\!/ a + \varepsilon\!\!\!/^* a^+)] =$

$[1+(g\not{K}/2pk)(\not{\varepsilon}a+\not{\varepsilon}^*a^+)]^{-1}$ from the left, we obtain

$$\left\{\not{p}-\not{K}\left[\left(1+\frac{g^2}{pk}\right)(a^+a+\tfrac{1}{2})+g\left(\frac{p\varepsilon}{pk}a+\frac{p\varepsilon^*}{pk}a^*\right)\right.\right.$$
$$\left.\left.-\frac{g^2}{2pk}(\varepsilon^2a^2+\varepsilon^{*2}a^{+2})\right]-\kappa+\frac{g^2}{2pk}\frac{1}{2}(\not{\varepsilon}\not{\varepsilon}^*-\not{\varepsilon}^*\not{\varepsilon})\not{K}\right\}\chi_p=0. \tag{3.9}$$

In this step we have achieved that all boson operators have a common matrix coefficient, and the diagonalisation of the boson part can now be carried out. At this point we should mention that in the previous section we reached the bispinor amplitudes (2.25d) and (2.27) in a straightforward manner, without introducing any ansatz. Though it is true that the ansätze (2.27) and (3.8) can be guessed from the form of the corresponding semiclassical wavefunction (Volkov state, see e.g. Bergou and Varró (1980)), we still felt it necessary to prove that the solution can be only of this form.

If ε is a real polarisation unit vector, then the boson part of (3.9) can be diagonalised in a way completely analogous to that of the previous section (details of this procedure will not be repeated here). In the case of circular polarisation, diagonalisation can be carried out in one step, since the (a^2+a^{+2}) term is missing due to $\varepsilon^2=\varepsilon^{*2}=0$. Let us define the displacement operator

$$D_\sigma=\exp(\sigma a^+-\sigma^*a) \tag{3.10}$$

having the (displacement) properties

$$D_\sigma^{-1}aD_\sigma=a+\sigma, \qquad D_\sigma^{-1}a^+D_\sigma=a^++\sigma^*. \tag{3.10a}$$

Application of the D_σ thus defined to (3.9) yields

$$\left\{\not{p}-\not{K}\left[\left(1+\frac{g^2}{pk}\right)(a^+a+\tfrac{1}{2}+|\sigma|^2+\sigma a^++\sigma^*a)+g\left(\frac{p\varepsilon}{pk}\sigma+\frac{p\varepsilon^*}{pk}\sigma^*\right)+g\left(\frac{p\varepsilon}{pk}a+\frac{p\varepsilon^*}{pk}a^+\right)\right]\right.$$
$$\left.+\frac{g^2}{2pk}\frac{1}{2}(\not{\varepsilon}\not{\varepsilon}^*-\not{\varepsilon}^*\not{\varepsilon})\not{k}-\kappa\right\}D_\sigma^{-1}\chi_p=0. \tag{3.11a}$$

If now $(1+g^2/pk)\sigma=-g(p\varepsilon^*/pk)$ then

$$\left[\not{p}-\not{K}\left(1+\frac{g^2}{pk}\right)(a^+a+\tfrac{1}{2}-|\sigma|^2)+\frac{g^2}{2pk}\frac{1}{2}(\not{\varepsilon}\not{\varepsilon}^*-\not{\varepsilon}^*\not{\varepsilon})-\kappa\right]D_\sigma^{-1}\chi_p=0. \tag{3.11b}$$

The solution of this equation is

$$D_\sigma^{-1}\chi_p=w|n\rangle, \qquad n=0,1,2,\ldots, \tag{3.11c}$$

where $|n\rangle$ is a number state of the EM mode and w is a constant bispinor satisfying the equation

$$[\not{p}-\not{K}(1+g^2/pk)(n+\tfrac{1}{2}-|\sigma|^2)+(g^2/2pk)\tfrac{1}{2}(\not{\varepsilon}\not{\varepsilon}^*-\not{\varepsilon}^*\not{\varepsilon})\not{K}-\kappa]w=0. \tag{3.11d}$$

This equation can be solved with the help of the projection method (Becker and Mitter 1974). As is proved in appendix 2 (see the relations (A2.9)–(A2.13)),

$$\tfrac{1}{2}(\not{\varepsilon}\not{\varepsilon}^*-\not{\varepsilon}^*\not{\varepsilon})\not{K}w=-\not{K}w. \tag{3.11e}$$

With the help of this relation equation (3.11d) can be written as

$$(\not{q}-\kappa)w=0 \tag{3.12a}$$

where

$$q = p - kf_p(n) \tag{3.12b}$$

and

$$f_p(n) \equiv (1 + g^2/pk)(n + \tfrac{1}{2} - |\sigma|^2) + g^2/2pk. \tag{3.12c}$$

Since $pk = qk$ and $p\varepsilon = q\varepsilon$ we have $f_p(n) = f_q(n)$, i.e.

$$p = q + f_q(n)k. \tag{3.13}$$

Equation (3.12a) is the well known free electron bispinor equation for w, the solution of which is

$$w = u_q. \tag{3.14}$$

With the help of equations (3.4), (3.8), (3.11c), (3.13) and (3.14), we obtain finally for the solution of (3.3)

$$\psi = \exp[ikx(a^+a + \tfrac{1}{2})][1 + (g\not{k}/2qk)(\not{\varepsilon}a + \not{\varepsilon}^*a^+)]u_q D_\sigma|n\rangle \exp\{-i[q + kf_q(n)]x\}. \tag{3.15}$$

The elimination technique outlined in the present and previous sections, and which consists of subsequent applications of certain unitary transformations, is of course a useful tool for finding the solutions of the corresponding Klein–Gordon equation as well. The representation-independent use of the boson operators makes it easy to survey the course of the solution. We also note here that the simplicity of our method gives the possibility of obtaining the propagators satisfying the corresponding inhomogeneous Klein–Gordon and Dirac equations in a tractable form. We shall leave this problem to a subsequent publication.

4. Some statistical aspects of the electron–one mode system

In the preceding sections we determined the stationary states of the system 'one electron + one quantised EM mode'. For circularly polarised photons, from (3.15)

$$\psi = \langle \boldsymbol{r}|\psi\rangle = \exp[-i\boldsymbol{kr}(a^+a + \tfrac{1}{2})][1 + (g/2qk)\not{k}(\not{\varepsilon}a + \not{\varepsilon}^*a^+)]u_q D_\sigma|n\rangle\, e^{-ipx} \tag{4.1}$$

where

$$p = q + k(1 + g^2/qk)(n + \tfrac{1}{2} - |\sigma|^2) + (g^2/2qk)k. \tag{4.1a}$$

For linearly polarised photons, from (2.28)

$$\psi = \langle \boldsymbol{r}|\psi\rangle = \exp[-i\boldsymbol{kr}(a^+a + \tfrac{1}{2})][1 + (g/2qk)\not{k}\not{\varepsilon}(a + a^+)]u_q C_\Theta D_\tau|n\rangle\, e^{-ipx} \tag{4.2}$$

where

$$p = q + k(1 + g^2/qk)(n + \tfrac{1}{2} - \tau^2)\,\mathrm{sech}\, 2\Theta. \tag{4.2a}$$

The total four-momentum of the system can be decomposed in a natural manner into a four-momentum q lying on the free mass shell, a four-momentum which is roughly an integer multiple of the four-vector k, and an intensity-dependent shift in the direction of k. Equation (4.1a), for example, can be written as

$$p = q + [n + \tfrac{1}{2} - (1 + g^2/qk)|\sigma|^2]k + (\kappa^2\nu^2/2qk)k \tag{4.3}$$

where

$$\nu^2 \equiv 2\kappa^{-2}g^2(n+1) = 2\alpha\lambda\rho\kappa^{-2}, \tag{4.3a}$$

ν^2 is a dimensionless intensity parameter, α is the fine structure constant, $\rho=(n+1)/V$ is the photon density and λ is the wavelength of radiation. The intensity parameter ν defined in (4.3a) is the same as the one introduced by Brown and Kibble (1964) in connection with the interpretation of the corresponding semiclassical state (Volkov state).

Since the interaction of the electron with the photons is taken into account exactly, we cannot say that $q+(\kappa^2\nu^2/2qk)k$ is the shifted four-momentum of the electron and n is the number of photons in the mode. We can merely say that this roughly holds for the expectation values of the corresponding quantities.

Due to the interaction, neither the electron nor the photon can be described by a pure state; thus we introduce the density operators of these subsystems defined as a partial trace of the total system's density operator over the other subsystem, i.e.

$$\rho_{\mathrm{e}}\equiv \mathrm{Tr}_{\mathrm{f}}\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle},\qquad \rho_{\mathrm{f}}\equiv \mathrm{Tr}_{\mathrm{e}}\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}. \tag{4.4}$$

Here Tr_{f} (Tr_{e}) denotes partial trace over the Hilbert space of the photons (electrons). In this section we shall investigate in more detail the circularly polarised case.

Taking the partial trace denoted in (4.4), with the help of the concrete wavefunction (4.1) we obtain for ρ_{e}

$$\rho_{\mathrm{e}}(\boldsymbol{r},\boldsymbol{r}')=\sum_{l=0}^{\infty} M_l(q,n)\exp\{\mathrm{i}[\boldsymbol{p}-(l+\tfrac{1}{2})\boldsymbol{k}](\boldsymbol{r}-\boldsymbol{r}')\} \tag{4.5}$$

where

$$M_l(q,n)=\frac{1}{\langle\psi|\psi\rangle}\left(c_l^{(0)}+\frac{g}{2qk}\not{K}(c_l^{(-)}\not{\varepsilon}+c_l^{(+)}\not{\varepsilon}^*)\right)u_q\bar{u}_q$$
$$\times\left(c_l^{(0)*}+\frac{g}{2qk}(c_l^{(+)*}\not{\varepsilon}+c_l^{(-)*}\not{\varepsilon}^*)\not{K}\right)\gamma_0. \tag{4.5a}$$

The coefficients c_l in (4.5a) are defined as

$$c_l^{(0)}=c_l^{(0)}(q,n)\equiv\langle l|D_\sigma|n\rangle,\qquad c_l^{(+)}=c_l^{(+)}(q,n)\equiv\langle l|a^+D_\sigma|n\rangle,$$
$$c_l^{(-)}=c_l^{(-)}(q,n)\equiv\langle l|aD_\sigma|n\rangle. \tag{4.5b}$$

The Fourier transform of (4.5) gives the components of ρ_{e} in momentum space:

$$\rho_{\mathrm{e}}(\boldsymbol{p}')=\sum_{l=0}^{\infty} M_l(q,n)\delta^{(3)}[\boldsymbol{p}-(l+\tfrac{1}{2})\boldsymbol{k}-\boldsymbol{p}']. \tag{4.6}$$

Consequently, the momentum distribution is given by $\mathrm{Tr}\,M_l(q,n)$. Since the system is in an eigenstate of the total momentum, the photon-number distribution is also given by the same matrix elements $\mathrm{Tr}\,M_l(q,n)$. This point will be discussed in more detail later on; here we confine ourselves to the qualitative statement that (4.6) describes the photon-induced level structure, which can also be derived from the corresponding semiclassical wavefunction. As we shall see, the quantities $\mathrm{Tr}\,M_l(q,n)$ in the case $n\gg 1$ coincide with the distribution obtained in external field approximation where the weighting factors are Bessel functions.

The coefficients c_l appearing in $M_l(q,n)$ and defined in (4.5b)—as has already been pointed out in I—are proportional to certain generalised Laguerre polynomials

(Abramowitz and Stegun 1964):

$$c_l(q,n)=\begin{cases}\exp(-|\sigma|^2/2)(n!\,l!)^{1/2}\sigma^{l-n}(1/l!)L_n^{(l-n)}(|\sigma|^2), & l>n,\\ \exp(-|\sigma|^2/2)(n!\,l!)^{1/2}(-\sigma^*)^{n-l}(1/n!)L_l^{(n-l)}(|\sigma|^2), & l<n.\end{cases} \tag{4.7}$$

The expectation value of the momentum of the electron is the difference between the total momentum and the expectation value of the momentum of the mode:

$$\langle\hat{\boldsymbol{p}}\rangle=\langle-\mathrm{i}\boldsymbol{\nabla}\rangle=\boldsymbol{p}-\boldsymbol{k}\langle a^+a+\tfrac{1}{2}\rangle. \tag{4.8}$$

From equations (4.8) and (4.1*a*) it is clear that the transverse components of q are equal to the transverse components of the expectation value of the momentum of the electron. The expectation value $\langle a^+a+\frac{1}{2}\rangle$ can be obtained by elementary algebra:

$$\begin{aligned}\langle a^+a+\tfrac{1}{2}\rangle=n+\tfrac{1}{2}+|\sigma|^2+[\![(p_0g/qk)\sigma E-k_0\{(2n+1)|\sigma|^2\\ +(2g/qk)[(2n+|\sigma|^2)-g^2/qk]\sigma E-(g^2/qk)(n+|\sigma|^2+1)\}]\!]\\ \times[p_0-k_0(n+\tfrac{1}{2}+|\sigma|^2)]^{-1}.\end{aligned} \tag{4.9}$$

The quantity

$$E\equiv\bar{u}_q\not{\varepsilon}\not{K}u_q \tag{4.9a}$$

depends on the helicity of the bispinor u_q, and is summarised in table 1.

Table 1. The values of $E\equiv\bar{u}_q\not{\varepsilon}\not{K}u_q$ for different electron helicities and photon polarisations (the three vectors $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$ and $\boldsymbol{k}$ form a right-handed frame). $\boldsymbol{n}=\boldsymbol{q}/|\boldsymbol{q}|$.

Photon polarisation \ Electron helicity	+1	−1
$\boldsymbol{\varepsilon}=(\boldsymbol{\varepsilon}_1+\mathrm{i}\boldsymbol{\varepsilon}_2)/\sqrt{2}$	$k_0(\boldsymbol{n\varepsilon})$	$-k_0(\boldsymbol{n\varepsilon})$
$\boldsymbol{\varepsilon}=(\boldsymbol{\varepsilon}_1-\mathrm{i}\boldsymbol{\varepsilon}_2)/\sqrt{2}$	$-k_0(\boldsymbol{n\varepsilon})$	$k_0(\boldsymbol{n\varepsilon})$

If we take the average of expression (4.9) for $\langle a^+a+\frac{1}{2}\rangle$ over the two possible helicity states of u_q, then the terms proportional to E vanish because the average of E over spin states is 0. Thus

$$\langle a^+a+\tfrac{1}{2}\rangle_{\mathrm{av}}=n+\tfrac{1}{2}+|\sigma|^2-\frac{k_0[(2n+1)|\sigma|^2-(g^2/qk)(n+|\sigma|^2+1)]}{p_0-k_0(n+\frac{1}{2}+|\sigma|^2)}. \tag{4.9b}$$

From here we may conclude that in the general case n cannot be identified with the expectation value of the number of photons in the mode. In some special cases, however, this identification can be carried through. If, for example, the transverse components of q are chosen to be zero, then $\sigma=0$ and from either (4.9) or (4.9*b*) we obtain the expectation value as

$$\langle a^+a\rangle=n+\delta,\qquad \delta=\frac{k_0\kappa^2\nu^2/2qk}{q_0+k_0\kappa^2\nu^2/2qk}. \tag{4.9c}$$

In the non-relativistic approximation ($\boldsymbol{qn}\ll\kappa$), $\delta\sim\nu^2/(2+\nu^2)$ and n means the number of photons in the mode to order $\sim\nu^2$. In the ultrarelativistic case ($|\boldsymbol{qn}|\gg\kappa$) $\delta=1$, i.e. δ satisfies the inequality

$$\nu^2/(\nu^2+2)<\delta<1. \tag{4.9d}$$

In the case of $\sigma = 0$ we have $D_\sigma = 1$, and instead of (4.1) ψ is given by the much simpler expression

$$\psi = \exp[-\mathrm{i}\boldsymbol{kr}(a^+a+\tfrac{1}{2})][1+(g/2qk)\not{K}(\not{\varepsilon}a+\not{\varepsilon}^*a^+)]u_q|n\rangle\,\mathrm{e}^{-\mathrm{i}px}. \tag{4.10}$$

Besides the photon number eigenstate $|n\rangle$, the excited states $|n\pm 1\rangle$ also appear in the wavefunction, due to the interaction of the electron spin with the electromagnetic mode. The spin may play an essential role at high intensities, as can be shown from the expectation value of the interaction energy of the electron–photon system:

$$\begin{aligned}\langle H_{\mathrm{int}}\rangle &= c\hbar\frac{2g\sigma(q\varepsilon)+2g^2(n+|\sigma|^2+1)}{p_0-k_0(n+\frac{1}{2}+|\sigma|^2)}\\ &= c\hbar\frac{-2(qk)|\sigma|^2+2g^2(n+1)}{p_0-k_0(n+\frac{1}{2}+|\sigma|^2)}.\end{aligned} \tag{4.11}$$

Here the contribution proportional to $2g^2(n+1+|\sigma|^2)$ arises from the spin. If the electron possesses longitudinal momentum only, then the interaction energy originates entirely from interaction of the spin with the EM field and is always positive. The transverse (perpendicular to $\boldsymbol{k}$) motion of the electron gives a negative contribution to the interaction energy (see the definition of σ in § 3). The sign of $\langle H_{\mathrm{int}}\rangle$ depends on the relative magnitude of these two opposite actions, i.e. the electron–photon interaction can be either attractive or repulsive. From (4.11) it can be seen that at high enough intensities ($2g^2(n+1)\equiv\kappa^2\nu^2 \gg 2(qk)|\sigma|^2$) the interaction energy comes almost entirely from the spin–radiation field interaction. This statement is further confirmed by the fact that in the cross sections of the multiphoton direct and inverse bremsstrahlung transitions of a free electron, the contribution arising from the spin part dominates at high enough intensities (Bergou and Varró 1980).

After this short digression we can easily evaluate the expression (4.8) of the electron momentum. From (4.9*b*) for $\sigma = 0$ we have

$$\langle -\mathrm{i}\hbar\nabla\rangle = \hbar[\boldsymbol{q}+(\kappa^2\nu^2/2qk)\boldsymbol{k}-\delta\boldsymbol{k}] \tag{4.12}$$

where δ is a positive quantity falling between the limits given by (4.9*d*). Thus the expectation value of the electron momentum is a free momentum containing an intensity-dependent shift. Let us consider now the other subsystem, i.e. the electromagnetic mode.

Carrying out the partial trace over the electron variables in the defining equation (4.4) of ρ_{f} yields

$$\rho_{\mathrm{f}}(l, l') \equiv \langle l|\rho_{\mathrm{f}}|l'\rangle = \rho_{\mathrm{f}}(l)\delta_{ll'}$$

where

$$\begin{aligned}\rho_{\mathrm{f}}(l) = \{&q_0|c_l^{(0)}|^2+(g/qk)\,\mathrm{Re}[c_l^{(0)}c_l^{(+)*}(k_0q\varepsilon+q_0E)+c_l^{(0)}c_l^{(-)*}(k_0q\varepsilon^*-q_0E)]\\ &+(k_0g^2/qk)|c_l^{(+)}|^2\}[p_0-k_0(n+\tfrac{1}{2}+|\sigma|^2)]^{-1}.\end{aligned} \tag{4.13}$$

Here E is defined according to (4.9*a*) and the coefficients $c_l^{(0)}$, $c_l^{(+)}$, $c_l^{(-)}$ are in accordance with (4.5*b*). In the following we shall use the average of $\rho_{\mathrm{f}}(l)$ over the helicity directions. We evaluate $\rho_{\mathrm{f}}(l)$ in the two limiting cases $n \gg 1$ and $n = 0$.

In the case of $n \gg 1$, from the definitions (4.5*b*) it is easy to derive the relations

$$c_l^{(+)} \approx (\sqrt{n}+\sigma^*)c_l^{(0)}, \qquad c_l^{(-)} \approx (\sqrt{n}+\sigma)c_l^{(0)}. \tag{4.13a}$$

Using this and the definition of σ in the expression for $\rho_f(l)$, we obtain

$$\rho_f(l) \approx |c_l^{(0)}|^2, \qquad n \gg 1. \tag{4.13b}$$

Besides (4.13*a*), the above approximation involves neglecting $d = [\mathrm{Re}(q\varepsilon)/(q_0 - \boldsymbol{nq})]\nu$ (if $q\varepsilon_1 = 0$ then $d = 0$). The result (4.13*b*) can be obtained formally from (4.13) by omitting the spin contributions, and one could then say that this contradicts the previous statement about the essential role played by the spin at high intensities. However, the conditions $|d| \ll 1$ and $n \gg 1$ are in contradiction only at $\nu \gtrsim 1$, i.e. at extremely high intensities. Substituting $c_l^{(0)}$ from equation (4.7) into (4.13*b*), the photon-number distribution can be expressed in terms of generalised Laguerre polynomials as

$$\rho_f(l) = \begin{cases} |\sigma|^{2(n-l)}(l!/n!)[L_l^{(n-l)}(|\sigma|^2)]^2 \exp(-|\sigma|^2), & l < n, \\ |\sigma|^{2(l-n)}(n!/l!)[L_n^{(l-n)}(|\sigma|^2)]^2 \exp(-|\sigma|^2), & l > n. \end{cases} \tag{4.14}$$

The mean value of the photon number is

$$\langle l \rangle = n + |\sigma|^2. \tag{4.14a}$$

The $|\sigma|^2$ contribution is usually negligible with respect to n. However, this is not the case with the mean-square deviation in the photon number, since it is proportional to both n and σ:

$$\langle l^2 \rangle - \langle l \rangle^2 = (2n+1)|\sigma|^2. \tag{4.14b}$$

As was pointed out in I, the distribution (4.14) can be replaced by the following symmetric distribution in the vicinity of n:

$$\rho_f(l) = J_{n-l}^2(z), \qquad n \gg 1, \; |n-l| \ll n, \tag{4.15}$$

where

$$z \equiv 2|\sigma|\sqrt{n}. \tag{4.15a}$$

We mention at this point that the same distribution (4.15) can be obtained from the Volkov solution to the Dirac equation containing a classical vector potential with amplitude

$$A_0 = 2g\sqrt{n}/(1 + g^2/qk).$$

The expectation value of the photon number is now n instead of $n + |\sigma|^2$ of the asymmetric distribution (4.14). Besides the special case $n \gg 1$, we can obtain an equally easy to interpret result for $\rho_f(l)$ in the opposite extreme $n = 0$ as well:

$$\rho_f(l) = P_l\left(1 + \frac{k_0[(g^2/qk)L_1^{(l)} \cos\varphi - (1 + g^2/qk)L_1^{(l)}]}{q_0 + k_0[g^2/qk - 2(1 + g^2/qk)|\sigma|^2]}\right) \tag{4.16}$$

where

$$P_l = |\sigma|^{2l} \exp(-|\sigma|^2)/l! \tag{4.16a}$$

is a Poisson distribution with parameter $|\sigma|^2$ and $\varphi = \mathrm{arc}\,\sigma$.

The expectation value of this distribution can be easily obtained from equation (4.9) with the substitution

$$\langle l \rangle = |\sigma|^2 + k_0 \frac{g^2/qk - (1 - g^2/qk)|\sigma|^2}{q_0 + k_0[g^2/qk - 2(1 + g^2/qk)|\sigma|^2]}. \tag{4.16b}$$

The fractional expression on the RHS of equation (4.16) is due to the spin–EM field interaction. If we neglect this term, we are left with the relativistic counterpart of the distribution obtained in I. This coincides with the photon statistics of one quantised mode interacting with a Klein–Gordon particle. Thus we see that the photon statistics arising from the lowest-energy state of the electron–photon system at fixed electron four-momentum q corresponds to a modified coherent state where the modification comes from the spin–mode interaction. Since in the $n = 0$ case the state (4.1) is the eigenfunction of the system 'electron + self field', the above statement can be formulated in a slightly different way, namely, that the self field of the electron is slightly different from a coherent state due to the spin–mode interaction.

In the linearly polarised case—as may be guessed from the structure of the factor $C_{\odot}D_{\tau}|n\rangle$ appearing in the state (4.2)—the photon statistics of the state parametrised by $n = 0$ is essentially different from the Poisson distribution. This difference arises from the appearance of the operator $C_{\odot}$ describing double-photon excitation processes. This point, however, will not be discussed any further here.

5. Compton scattering

In this section we shall study the interaction of a free electron with two quantised modes of the radiation field. One of the modes will be taken into account exactly, whereas the other one will be treated by perturbation theory. This means that we shall consider nonlinear Compton scattering when the amount of photons in the mode characterised by polarisation vector ε and wavevector k may change by an arbitrary number, while one photon with polarisation ε' and wavevector k' is being emitted or absorbed.

Let us consider, for example, the transition amplitude for emission

$$T_{\mathrm{fi}} = -\mathrm{i}g' \int \mathrm{d}^4x\, \bar{\psi}_{\mathrm{f}} \not{\varepsilon}' \exp(\mathrm{i}k'x)\psi_{\mathrm{i}} \tag{5.1}$$

where

$$g' \equiv (\alpha\lambda' V^{-1})^{1/2} \tag{5.1a}$$

is defined analogously to g.

In (5.1) ψ_{f} and ψ_{i} are states of the type (4.1) or (4.2) parametrised by quantum numbers $q'n'$ and qn respectively, i.e. the effect of the mode $\{k, \varepsilon\}$ is taken into account up to infinite order in them. As we have already seen in the previous section, q and n may be approximately identified with the expectation value of the four-momentum of the electron and expectation value of the photon number in the mode $\{k, \varepsilon\}$ respectively. Thus, the amplitude (5.1) describes a process where $n - n'$ photons are scattered by the electron having initially four-momentum q, and the appearing 'new-photon' is parametrised by k', ε'. After performing the integration over the four-space, we obtain from (5.1) the result (normalisation factors of the initial and final states are omitted)

$$T_{\mathrm{fi}} = -\mathrm{i}g'(2\pi)^4\delta^{(4)}(p' + k' - p)\bar{\Phi}_{p'}\not{\varepsilon}'\Phi_p. \tag{5.2}$$

Here $\Phi_{p'}$ and Φ_p are to be taken from equations (3.4)–(3.7).

The Dirac delta function on the RHS of equation (5.2)—taking into account the form of p as given by (4.1a) and (4.2a) respectively—expresses the conservation law for the linearly polarised mode

$$p' + k' = p$$

or

$$q'+k(1+g^2/q'k)(n'+\tfrac{1}{2}-\tau'^2)\operatorname{sech}2\Theta'+k'=q+k(1+g^2/qk)(n+\tfrac{1}{2}-\tau^2)\operatorname{sech}2\Theta \tag{5.3}$$

and for the circularly polarised mode

$$p'+k'=p$$

or

$$q'+k[(1+g^2/q'k)(n'+\tfrac{1}{2}-|\sigma'|^2)+g^2/2q'k]+k'$$
$$=q+k[(1+g^2/qk)(n+\tfrac{1}{2}-|\sigma|^2)+g^2/2qk]. \tag{5.4}$$

The parameters Θ and τ in (5.3) are defined by equations (2.15a) and (2.18) respectively, and the parameter σ in (5.4) is defined by (3.11a). The corresponding parameters of the final state are denoted by primed quantities.

Here, for the sake of simplicity, we shall deal only with the detailed evaluation of (5.4). In the case of an electron which is initially at rest in the average ($q=0$, whence $\sigma=0$), we obtain the following quadratic equation from (5.4) for the wavenumber k_0' of the emitted photon:

$$\kappa k_0'+k_0'(1-\cos\theta)(k_0 l+\kappa\nu_n^2)=\kappa(k_0 l+\kappa\nu_n^2\delta)+g^2\frac{k_0^2\sin^2\theta}{\kappa k_0-k_0k_0'(1-\cos\theta)+g^2} \tag{5.4a}$$

where

$$l\equiv n-n'>0, \qquad \nu_n^2\equiv(g^2/\kappa^2)(n+\tfrac{1}{2})=(2\pi)^{-2}\alpha\rho\lambda\lambda_c^2, \tag{5.4b}$$

θ is the scattering angle (the angle between the wavevectors of the incoming and scattered photons), $\rho=(n+\frac{1}{2})/V$ is the initial photon density and $\lambda_c=h/mc$ is the Compton wavelength. We furthermore introduced the parameter

$$\delta=(n-n')/n \tag{5.4c}$$

which describes the relative depletion.

The solution of equation (5.4a) is

$$k_0'=[lk_0(1-\cos\theta)+\beta]\pm\kappa\{[1+\nu_n^2(1-\cos\theta)]^2-4\nu_n^2\delta\sin^2\theta\}^{1/2}$$
$$\times\left[2\left(\frac{\beta k_0(1-\cos\theta)+g^2\sin^2\theta}{\kappa k_0+g^2}\right)\right]^{-1} \tag{5.5}$$

where

$$\beta\equiv\kappa+(lk_0+\kappa\nu_n^2)(1-\cos\theta). \tag{5.5a}$$

If $\nu_{n'}^2\approx\nu_n^2\equiv\nu^2$, i.e. if we neglect the change in the intensity parameter and omit the terms $g^2\sin^2\theta$ and g^2 in the denominator of (5.5), we obtain two values for $\omega'\equiv ck_0'$,

$$\omega_+'=\omega_c/(1-\cos\theta) \tag{5.5b}$$

and

$$\omega_-'=\frac{l\omega}{1+[(l\omega/\omega_c)+\nu^2](1-\cos\theta)}. \tag{5.5c}$$

We note that ω_-' gives ω_+' as a limiting case (Goldman 1964)

$$\lim_{l\to\infty}\omega_-'=\omega_+'. \tag{5.5d}$$

From the semiclassical description of the very same problem, one can obtain an expression in complete analogy with (5.5c) (Brown and Kibble 1964). If we neglect the second term on the RHS of (5.4a), then we are left with a linear equation for k'_0 yielding the solution for $\omega' \equiv ck'_0$

$$\omega' = \frac{l\omega + \omega_c \nu^2 \delta}{1 + [(l\omega/\omega_c) + \nu^2](1 - \cos\theta)}. \tag{5.6}$$

According to (5.6), if we do not neglect depletion of the mode, then there is a contribution proportional to $\omega_c \nu^2 \delta$ in addition to the usual frequency shift. If, for example, $\nu^2 \sim 1$ and $\omega \approx 10^{-6}\omega_c$ (optical frequency) and the parameter δ of the relative depletion is 10^{-6}, then $\omega' \approx (l+1)\omega/[1 + (l \times 10^{-6} + 1)(1 - \cos\theta)]$, i.e. in the scattering process of l photons, instead of the lth harmonics ($\omega' = l\omega$) the next-order ($\omega' \approx (l+1)\omega$) harmonic appears. Thus, the effect of depletion can be substantial in certain cases.

After this analysis of the frequency condition of the nonlinear Compton scattering, let us proceed now to the evaluation of the transition amplitude. The matrix elements $\Phi_{p'}\not{\varepsilon}\Phi_p$ appearing in (5.2) have the form

$$\bar{\Phi}_{p'}\not{\varepsilon}\Phi_p = \begin{cases} \bar{u}_{q's'}\mathcal{M}_{\mathrm{L}} u_{qs} \\ \bar{u}_{q's'}\mathcal{M}_{\mathrm{C}} u_{qs} \end{cases} \tag{5.7a}$$

where

$$\mathcal{M}_{\mathrm{L}} \equiv \langle n'|D_{\tau'}^{-1} C_{\Theta'}^{-1}\Big(\not{\varepsilon}' + \frac{g}{2q'k}(a + a^+)\not{\varepsilon}\not{k}\not{\varepsilon}'$$
$$+ \frac{g}{2qk}(a + a^+)\not{\varepsilon}'\not{k}\not{\varepsilon} + \frac{g^2}{4(qk)(q'k)}(a + a^+)^2\not{\varepsilon}\not{k}\not{\varepsilon}'\not{k}\not{\varepsilon}\Big) C_\Theta D_\tau|n\rangle \tag{5.7b}$$

in the case of a linearly polarised mode $\{k, \varepsilon\}$ and

$$\mathcal{M}_{\mathrm{C}} \equiv \langle n'|D_{\sigma'}^{-1}\Big(\not{\varepsilon}' + \frac{g}{2q'k}(\not{\varepsilon}a + \not{\varepsilon}^* a^+)\not{k}\not{\varepsilon}' + \frac{g}{2qk}\not{\varepsilon}'\not{k}(\not{\varepsilon}a + \not{\varepsilon}^* a^+)$$
$$+ \frac{g^2}{4(qk)(q'k)}(\not{\varepsilon}a + \not{\varepsilon}^* a^+)\not{k}\not{\varepsilon}'\not{k}(\not{\varepsilon}a + \not{\varepsilon}^* a^+)\Big) D_\sigma|n\rangle \tag{5.7c}$$

in the case of a circularly polarised mode.

The transition probability of the scattering process is proportional to the quantity $|\bar{u}_{q's'}\mathcal{M}u_{qs}|^2$. The evaluation of this quantity is greatly facilitated if we notice that $\bar{\Phi}_{p'}\not{k}'\Phi_p = 0$. From the conservation law (5.4) we have $p' + k' = p$, and using equation (3.7) yields

$$\bar{\Phi}_{p'}\not{k}'\Phi_p = \bar{\Phi}_{p'}(\not{p} - \not{p}')\Phi_p = 0.$$

From here it follows that the matrix element $\bar{\Phi}_{p'}\not{\varepsilon}\Phi_p$ is invariant under the substitution $\varepsilon' \to \varepsilon'' \equiv \varepsilon' + uk'$ where u is an arbitrary c-number. It is convenient to choose $u = -(k\varepsilon'/kk')$ since then

$$k\varepsilon'' = 0 \qquad \text{where} \qquad \varepsilon'' = \varepsilon' - (k\varepsilon'/kk')k'. \tag{5.8}$$

With the help of this relationship

$$\bar{\Phi}_{p'}\not{\varepsilon}'\Phi_p = \begin{cases} \bar{u}_{q's'}M_{\mathrm{L}} u_{qs} \\ \bar{u}_{q's'}M_{\mathrm{C}} u_{qs} \end{cases} \tag{5.9a}$$

where

$$M_{\rm L}\equiv A_0\not{\varepsilon}''+(g/2q'k)A_1\not{\varepsilon}\not{K}\not{\varepsilon}''+(g/2qk)A_1\not{\varepsilon}''\not{K}\not{\varepsilon}, \tag{5.9b}$$

$$A_i\equiv\langle n|D_{\tau'}^{-1}C_{\Theta}^{-1}(a+a^+)^iC_{\Theta}D_\tau|n\rangle \qquad (i=0,1) \tag{5.9c}$$

and

$$M_{\rm C}\equiv B_0\not{\varepsilon}''+(g/2q'k)(\not{\varepsilon}B_-+\not{\varepsilon}^*B_+)\not{K}\not{\varepsilon}''+(g/2qk)\not{\varepsilon}''\not{K}(\not{\varepsilon}B_-+\not{\varepsilon}^*B_+), \tag{5.9d}$$

$$B_0\equiv\langle n'|D_{\sigma'}^{-1}D_\sigma|n\rangle, \qquad B_-\equiv\langle n'|D_{\sigma'}^{-1}aD_\sigma|n\rangle, \qquad B_+\equiv\langle n'|D_{\sigma'}^{-1}a^+D_\sigma|n\rangle. \tag{5.9e}$$

We average over initial and sum over final spin states in the usual manner in the expression $|\bar{u}_{q's'}Mu_{qs}|^2$, yielding for linearly polarised photons (with the normalisation $\bar{u}u=1$)

$$\begin{aligned}|t_{\rm fi}|^2_{\rm av}&\equiv\tfrac{1}{2}\sum_{s,s'}|\bar{u}_{q's'}M_{\rm L}u_{qs}|^2\\ &=\frac{1}{2\kappa^2}\bigg\{[2(q\varepsilon'')^2+(qq')-\kappa^2]|A_0|^2\\ &\quad+g\bigg[(kk')\bigg(\frac{q'\varepsilon}{q'k}-\frac{q\varepsilon}{qk}\bigg)-4(q\varepsilon'')(\varepsilon\varepsilon'')\bigg]\,{\rm Re}\,A_0^*A_1\\ &\quad+g^2\bigg(\frac{(kk')^2}{2(qk)(q'k)}+2(\varepsilon\varepsilon'')^2\bigg)|A_1|^2\bigg\}\end{aligned} \tag{5.10}$$

and for circularly polarised photons

$$\begin{aligned}|t_{\rm fi}|^2_{\rm av}&\equiv\tfrac{1}{2}\sum_{s,s'}|\bar{u}_{q's'}M_{\rm C}u_{qs}|^2\\ &=\frac{1}{2\kappa^2}\bigg\{[2(q\varepsilon'')^2+(qq')-\kappa^2]|B_0|^2\\ &\quad+g\,{\rm Re}\bigg[(kk')\bigg(\frac{q'\varepsilon}{q'k}-\frac{q\varepsilon}{qk}\bigg)-4(q\varepsilon'')(\varepsilon\varepsilon'')\bigg](B_0^*B_-+B_0B_+^*)\\ &\quad+g^2\bigg(\frac{(kk')^2}{2(qk)(q'k)}+2|\varepsilon\varepsilon''|^2\bigg)(|B_-|^2+|B_+|^2)+4g^2\,{\rm Re}\,B_-B_+(\varepsilon\varepsilon'')^2\bigg\}\end{aligned} \tag{5.11}$$

(Re denotes real part). We note that (5.10) is formally identical to the result obtained by Nikishov and Ritus (1964) in the semiclassical description of the same problem.

Equation (5.11), which is valid for circularly polarised photons, can be brought to the form

$$|t_{\rm fi}|^2_{\rm av}=|C|^2+(kk'/2\kappa^2)|D|^2 \tag{5.11a}$$

where

$$C\equiv(1/\kappa)[qB_0-g(\varepsilon B_-+\varepsilon^*B_+)]\varepsilon'' \tag{5.11b}$$

and

$$\begin{aligned}|D|^2&\equiv[n-n'-(\zeta-\xi)]|B_0|^2+\tfrac{1}{2}g^2(1/q'k-1/qk)(|B_-|^2+|B_+|^2)\\ &\quad+g\,{\rm Re}[(q'\varepsilon/q'k)(B_0^*B_-+B_0B_+^*)].\end{aligned} \tag{5.11c}$$

The meaning of the parameters ζ and ξ in (5.11c) is given by

$$\zeta \equiv (g^2/q'k)(n'+1)-(g^2/qk)(n+1) \tag{5.11d}$$

and

$$\xi \equiv g^2 \frac{|q'\varepsilon|^2}{(q'k)(q'k+g^2)} - g^2 \frac{|q\varepsilon|^2}{(qk)(qk+g^2)}. \tag{5.11e}$$

Equations (5.11a)–(5.11d) are formally in complete analogy with the solution of the corresponding semiclassical problem (Brown and Kibble 1964). Let us investigate now the behaviour of the quantities B defined in equation (5.9e). We assume that the electrons are at rest in the average before scattering ($\boldsymbol{q} = 0$ so that $\sigma = 0$). Application of the Baker–Hausdorff theorem (eg Messiah 1964) leads to

$$B_0 = \langle n'|D_{\sigma'}^{-1}|n\rangle = \langle n'|\exp(\sigma' a^+)\exp(-\sigma'^* a)|n\rangle \exp(-\tfrac{1}{2}|\sigma'|^2), \tag{5.12a}$$

$$B_- = \langle n'|D_{\sigma'}^{-1} a|n\rangle = n^{1/2}\langle n'|\exp(\sigma' a^+)\exp(-\sigma'^* a)|n-1\rangle \exp(-\tfrac{1}{2}|\sigma'|^2), \tag{5.12b}$$

$$B_+ = \langle n'|D_{\sigma'}^{-1} a^+|n\rangle = (n+1)^{1/2}\langle n'|\exp(\sigma' a^+)\exp(-\sigma'^* a)|n+1\rangle \exp(-\tfrac{1}{2}|\sigma'|^2). \tag{5.12c}$$

It can be verified by elementary algebra that

$$\langle n'|\exp(\sigma' a^+)\exp(-\sigma'^* a)|n\rangle$$
$$= \begin{cases} (n'!\,n!)^{1/2}(\sigma')^{n'-n} \displaystyle\sum_{l=0}^{n} \frac{(-|\sigma'|^2)^l}{(n'-n+l)!\,(n-l)!\,l!}, & n'>n, \\ (n'!\,n!)^{1/2}(-\sigma'^*)^{n-n'} \displaystyle\sum_{l=0}^{n'} \frac{(-|\sigma'|^2)^l}{(n-n'+l)!\,(n'-l)!\,l!}, & n>n'. \end{cases} \tag{5.12d}$$

With the help of this last expression we can easily show that the B coefficients are proportional to certain generalised Laguerre polynomials (Abramowitz and Stegun 1964), namely, by using (5.12a) and (5.12d)

$$B_0 = \begin{cases} (n'!/n!)^{1/2}(-\sigma'^*)^{n-n'} L_{n'}^{(n-n')}(|\sigma'|^2)\exp(-\tfrac{1}{2}|\sigma'|^2), & n'<n, \\ (n!/n'!)^{1/2}(\sigma')^{n'-n} L_{n}^{(n'-n)}(|\sigma'|^2)\exp(-\tfrac{1}{2}|\sigma'|^2), & n<n'. \end{cases} \tag{5.12e}$$

B_+ and B_- have similar expressions in terms of the Laguerre polynomials $L_{n'}^{(n-1-n')}$ and $L_{n'}^{(n+1-n')}$. It can also be shown that if $n' = n_0 \pm n$ and $n_0 \gg 1$ and $n_0 \gg n$, then the RHS of (5.12d) can be approximated by Bessel functions of integer order (Abramowitz and Stegun 1964) as

$$\langle n_0 \pm n|\exp(\sigma' a^+)\exp(-\sigma'^* a)|n_0\rangle \approx J_{\pm n}(2|\sigma'|n_0^{1/2})\exp(\pm \mathrm{i} n\varphi). \tag{5.12f}$$

If in the argument $2g(q'\varepsilon/q'k)n_0^{1/2}$ of the Bessel functions we identify $2gn_0^{1/2}$ with the amplitude of the vector potential of the corresponding classical radiation field, then the transition amplitudes (5.11) are in full agreement with the semiclassical amplitudes obtained from Volkov states (Brown and Kibble 1964). Thus, not very surprisingly, we are led to the conclusion that if the initial expectation value of the photon number in the mode $\{k, \varepsilon\}$ is large, and if the relative depletion of the mode during scattering is negligible ($n \ll n_0$), then our results reproduce the semiclassical results.

Let us consider therefore such processes where the depletion of the mode is maximal ($\delta = 1$), i.e. the expectation value of the photon number in the scattering mode decreases from n to 0. In the case $n = 1$ we can easily obtain the matrix elements (5.11) with the help of the conservation law (5.4) and frequency condition (5.6):

$$|t_{\mathrm{fi}}^{(1)}|_{\mathrm{av}}^2=\left(\frac{g}{\kappa}\right)^2\frac{1}{4}\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-2+4|\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'|^2\right)\exp\left[-\left(\frac{g}{\kappa}\right)^2|\hat{\boldsymbol{k}}'\boldsymbol{\varepsilon}|^2\right]. \tag{5.13}$$

Here $\hat{\boldsymbol{k}}'$ is a unit vector in the direction of the scattered wave. Apart from the exponential factor, (5.13) is the well known Klein–Nishina formula. In computing (5.13) we kept only terms of the order of g^2/κ^2 in $|t_{\mathrm{fi}}^{(1)}|^2$, since usually $g^2/\kappa^2 \ll 1$ (g^2/κ^2 is the intensity parameter corresponding to photon density $1/V$). We also note that in the expression (5.11*b*) for C the terms of the order g^2/κ^2 containing $(q\varepsilon'')B_0$ and $(k'\varepsilon)B_-$ cancel. Similarly, in the expression (5.11*c*) for $|D|^2$ the terms $|B_0|^2$ and $g\,\mathrm{Re}(q'\varepsilon/q'k)B_0^*B_-$ cancel. It can be shown that the same is true for the transition matrix elements of an arbitrary nth-order ($n\to 0$ type) process. The transition probability of the two-photon Compton scattering can be calculated quite similarly to the previous one from equation (5.11):

$$|t_{\mathrm{fi}}^{(2)}|_{\mathrm{av}}^2=\left(\frac{2g}{\kappa}\right)^4\frac{1}{4}\left(\frac{2\omega}{\omega'}+\frac{\omega'}{2\omega}-2+4|\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'|^2\right)\tfrac{1}{2}|\hat{\boldsymbol{k}}'\boldsymbol{\varepsilon}|^2\exp\left[-\left(\frac{2g}{\kappa}\right)^2|\hat{\boldsymbol{k}}'\boldsymbol{\varepsilon}|^2\right]. \tag{5.14}$$

Here we should mention that in deriving (5.13) and (5.14) we have neglected the intensity-dependent shift in the frequency equation (5.6). In the general case, introducing the above simplifying assumptions, the following formula is valid:

$$|t_{\mathrm{fi}}^{(n)}|_{\mathrm{av}}^2=\frac{1}{4}\left(\frac{n\omega}{\omega'}+\frac{\omega'}{n\omega}-2+4|\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'|^2\right)\frac{(gn/\kappa)^{2n}}{n!}|\hat{\boldsymbol{k}}'\boldsymbol{\varepsilon}|^{2(n-1)}\exp\left[-\left(\frac{gn}{\kappa}\right)^2|\hat{\boldsymbol{k}}'\boldsymbol{\varepsilon}|^2\right]. \tag{5.15}$$

The transition probability of the nth-order nonlinear Compton scattering thus differs from the usual value $\frac{1}{4}(n\omega/\omega'+\omega'/n\omega-2+4|\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'|^2)$ by the factor

$$[(\mu^2n^2)^n/n!]\exp(-\mu^2n^2), \qquad \mu\equiv(g/\kappa)|\hat{\boldsymbol{k}}'\boldsymbol{\varepsilon}|. \tag{5.16}$$

This factor depends on the order of the process. The study of processes more general than the ones of type $n\to 0$ is beyond the scope of the present work.

6. Summary and discussion

In paper I we set forth that for the description of nonlinear transitions of a free electron in an intense field semiclassical methods are not always applicable. This is the case especially if the depletion is comparable to the initial number of photons in the intense mode. A perturbation theoretical treatment of such highly nonlinear processes using the full QED formalism is, on the other hand, extremely complicated. Therefore in I we worked out a relatively simple method to handle such problems. That method is developed further in the present paper, inasmuch as it is generalised to the relativistic case and the dipole approximation is dropped. In § 5 we illustrated, using the nonlinear Compton scattering (harmonic generation), how one can determine in one step the amplitude of transition for the process of absorption of n photons from the intense mode and simultaneous appearance of one photon in the scattering mode. To perform this, in §§ 2 and 3 we solved exactly the Dirac equation of the system 'electron + one quantised EM mode' for linear and circular polarisation. The solutions are given by equations (2.28) and (3.15). With the help of these solutions, we have calculated analytically the cross section of the nonlinear Compton scattering. We have explicitly shown that in the limit of large initial photon number and small depletion, equation (5.11) reduces to the semiclassical results obtained with the help of the Volkov solution (Alperin 1944, Brown and Kibble 1964). In the other extreme case of complete depletion we also obtained a closed-form result, equation (5.15).

This result can be regarded as the generalisation of the semiclassical result to the large-depletion case. It contains the depletion factor (5.16), and so it takes into account explicitly the depletion of the intense mode due to the highly nonlinear scattering process.

In both cases (linear and circular polarisation) we have found that the spectrum of the full system 'electron + quantised mode' belonging to stationary states can be parametrised by two quantum numbers p and n. The continuous parameter p specifying the total four-momentum of the system can be decomposed in a natural manner into a four-momentum q lying on the free mass-shell, a four-momentum which is roughly an integer multiple of the four-momentum k of the free mode and an intensity-dependent shift in the direction of k (see equation (4.3)). Thus, the total four-momentum—though it is a good quantum number—is not simply a sum of the four-momenta of the electron (q) and EM mode (nk), but, in addition, contains correction terms arising from the interaction. The discreteness of the parameter n is related to the fact that one of the component subsystems (the EM mode) can be excited in discrete steps, though in the interacting case n cannot be identified with the photon content in the mode (excitation level of the mode) in the strict sense of the word. This point is discussed in more detail together with the statistical aspects of the problem in § 4. Here we briefly recall just one interesting property of the photon distribution function, namely that the distribution corresponding to the lowest-energy state of the full system at fixed electron four-momentum q describes a slightly modified coherent state which is, in fact, very similar to the coherent state found in I. The slight modification is due to the spin–mode interaction.

Finally, we mention that the Dirac equation in a classical external field was first solved exactly by Volkov (1935). An independent solution was obtained and applied to nonlinear Compton scattering by Alperin (1944). The exact solution of the eigenvalue problem and stationary states of the system 'Dirac electron + one quantised EM mode' was given for the linearly polarised case by Berson (1969) using the coordinate representation for the boson operators. In the present paper we reobtained this solution in a representation-independent manner by purely algebraic methods. This solution can be immediately obtained by the usual Volkov ansatz (2.27) for the bispinor amplitude but, as we have demonstrated, there is no need to introduce this ansatz since this form of the solution derives automatically in the Majorana representation. The solution for the circularly polarised case was made possible by using the projection technique (Neville and Rohrlich 1971a, b, Becker and Mitter 1974). For the sake of brevity in this case we introduced the Volkov ansatz in equation (3.8) instead of a detailed algebraic derivation.

Essentially similar problems were treated in a series of papers (Fried and Eberly 1964, Eberly and Reiss 1966, Reiss and Eberly 1966, Eberly 1969), where instead of stationary states the Green function of the system was calculated. An almost exact summation of the diagrams for the Green function was carried out. By 'almost', we mean that the photon number in the intense mode was the same fixed value in diagrams of different order, i.e. depletion was neglected. Here we also note that our method is applicable to the Green function problem, yielding a simple algebraic solution to it which also describes depletion of the mode. This point is left, however, to a future publication.

We have also shown how our results reduce to the semiclassical ones in the limit of large initial photon number and small depletion. Nonlinear Compton scattering was investigated by semiclassical methods in a number of papers (Goldman 1964, Brown and Kibble 1964, Nikishov and Ritus 1964). Closed-form analytical results, equation (5.15), are also given for the other limiting case, i.e. for complete depletion. The depletion factor (5.16) ensures the convergence of the sum of different high-order contributions, thus giving credit to the power of the method outlined here and in I.

Appendix 1. Notation

The scalar product of two four-vectors $a=\{a^\mu\}\equiv(a_0, \boldsymbol{a})$ and $b=\{b^\mu\}=(b_0, \boldsymbol{b})$ is defined as

$$ab = g_{\mu\nu}a^\mu b^\nu = a^\mu b_\mu = a_0 b_0 - \boldsymbol{ab}. \tag{A1.1}$$

Here

$$g_{\mu\nu}=0 \qquad \text{if } \mu\neq\nu \ (\mu, \nu=0, 1, 2, 3)$$

and

$$g_{00}=-g_{ii}=1 \qquad (i=1, 2, 3).$$

The definition of the four-gradient is

$$\partial=\{\partial_\mu\}, \qquad \partial_\mu=\partial/\partial x^\mu, \qquad \text{where } \{x^\mu\}=(ct, \boldsymbol{x}). \tag{A1.2}$$

In § 2 we used concrete forms of the $\boldsymbol{\alpha}$ and β matrices. In standard representation

$$\alpha_{x,y,z}=\begin{pmatrix} 0 & \sigma_{x,y,z} \\ \sigma_{x,y,z} & 0 \end{pmatrix}, \qquad \beta=\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{A1.3}$$

Here $\sigma_{x,y,z}$ are the 2×2 Pauli matrices and $\mathbb{1}$ stands for the 2×2 unit matrix. The Dirac γ matrices are defined as

$$\gamma^0=\beta, \qquad \gamma^{1,2,3}=\beta\alpha_{x,y,z}. \tag{A1.4}$$

The γ matrices satisfy the relations

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}.$$

Throughout the present paper we use the Feynman dagger (or slash) notation for scalar products of the type $a\gamma=a^\mu\gamma_\mu$, i.e.

$$a\gamma=\not{a}. \tag{A1.5}$$

We make the transition from the standard representation of $\boldsymbol{\alpha}$ and β to the Majorana representation with the help of the unitary matrix

$$U^{(\mathrm{M})}\equiv(1/\sqrt{2})(\alpha_y+\beta). \tag{A1.6}$$

In this representation

$$U^{(\mathrm{M})}\alpha_x U^{(\mathrm{M})-1}=-\alpha_x, \qquad U^{(\mathrm{M})}\alpha_y U^{(\mathrm{M})-1}=\beta, \tag{A1.6a}$$

$$U^{(\mathrm{M})}\alpha_z U^{(\mathrm{M})-1}=-\alpha_z, \qquad U^{(\mathrm{M})}\beta U^{(\mathrm{M})-1}=\alpha_y. \tag{A1.6b}$$

Appendix 2. Light-like formalism and the projection technique

The light-like formalism is based on the fact that the null vectors

$$n=\{n^{\mu}\}\equiv\frac{c}{\sqrt{2}\omega}\{k^{\mu}\}=\frac{1}{\sqrt{2}}(1,\boldsymbol{n}),\qquad \hat{n}=\{\hat{n}^{\mu}\}\equiv\frac{1}{\sqrt{2}}(1,-\boldsymbol{n}),\tag{A2.1a}$$

together with the space-like unit vectors

$$\varepsilon_i\equiv(0,\boldsymbol{\varepsilon}_i),\qquad i=1,2,\qquad \boldsymbol{\varepsilon}_1\perp\boldsymbol{\varepsilon}_2,\tag{A2.1b}$$

form a complete set in Minkowski space. This complete set is used as a basis for the decomposition of the four-vectors (Neville and Rohrlich 1971a, b). The light-like components of an arbitrary four-vector a are defined as

$$a=na_u+\hat{n}a_v-e_ia_i\qquad(\text{summation over } i=1,2)\tag{A2.2}$$

where

$$a_u\equiv\hat{n}a,\qquad a_v\equiv na,\qquad a_i\equiv-e_ia.\tag{A2.2a}$$

The scalar product of two four-vectors a and b can be given in terms of light-like components as

$$ab=a_ub_v+a_vb_u-a_ib_i.\tag{A2.3}$$

For the light-like components of $\{\gamma^{\mu}\}$ it can easily be shown that

$$\begin{aligned}&\gamma_u^2=\gamma_v^2=0,\qquad \gamma_u\gamma_i=-\gamma_i\gamma_u,\qquad \gamma_v\gamma_i=-\gamma_i\gamma_v,\\ &\gamma_u\gamma_v+\gamma_v\gamma_u=2,\qquad \gamma_u\not{g}=-\not{g}\gamma_u,\qquad \gamma_v\not{g}=-\not{g}\gamma_v.\end{aligned}\tag{A2.4}$$

Let us define the projection operators $\mathbb{P}_u$ and $\mathbb{P}_v$ and the bispinors w_u and w_v as

$$\mathbb{P}_u\equiv\tfrac{1}{2}\gamma_v\gamma_u,\qquad \mathbb{P}_v\equiv\tfrac{1}{2}\gamma_u\gamma_v,\tag{A2.5}$$

$$w_u\equiv\gamma_uw,\qquad w_v\equiv\gamma_vw.\tag{A2.6}$$

Using the properties (A2.4), the following relations can be shown to be valid:

$$\mathbb{P}_uw_u=\mathbb{P}_vw_v=0,\qquad \mathbb{P}_uw_v=w_v,\qquad \mathbb{P}_vw_u=w_u.\tag{A2.7}$$

After these preliminaries we proceed to the solution of equation (3.10d). Let us introduce the four-vector

$$p'=p-k(1+g^2/pk)(n+\tfrac{1}{2}-|\sigma|^2);\tag{A2.8}$$

then by taking into account equation (A2.6), equation (3.10d) can be written in light-like components as

$$p'_uw_v+p'_vw_u-(p'_i\gamma_i+\kappa)w+(g^2/2p'_v)\tfrac{1}{2}(\not{g}\not{g}^*-\not{g}^*\not{g})w_v=0.\tag{A2.9}$$

(Here we have made use of $p_v=p'_v$.)

If we multiply (A2.9) by $\mathbb{P}_v$ from the left and take into account the projection properties (A2.7), then we find the following simple connection between w_u and w_v:

$$w_u=[(p'_i\gamma_i+\kappa)/2p'_v]\gamma_uw_v.\tag{A2.10}$$

On the other hand, by multiplying (A2.9) by $\mathbb{P}_u$ from the left, we obtain

$$p'_uw_v-(p'_i\gamma_i+\kappa)\tfrac{1}{2}\gamma_vw_u+(g^2/2p'_v)\tfrac{1}{2}(\not{g}\not{g}^*-\not{g}^*\not{g})w_v=0.\tag{A2.11}$$

Substitution of w_u from (A2.10) into (A2.11) yields

$$[\Delta + g^2 \tfrac{1}{2}(\not{g}\not{g}^* - \not{g}^*\not{g})]w_v = 0 \tag{A2.12}$$

where

$$\Delta \equiv 2p'_u p'_v - p'_i p'_i - \kappa^2 = p'^2 - \kappa^2. \tag{A2.12a}$$

We see from equation (A2.12) that

$$g^2 \tfrac{1}{2}(\not{g}\not{g}^* - \not{g}^*\not{g})\not{K}w = -\Delta \not{K}w. \tag{A2.12b}$$

Upon multiplication by $[\Delta - g^2 \frac{1}{2}(\not{g}\not{g}^* - \not{g}^*\not{g})]$ the solution of the eigenvalue equation (A2.12) is derived automatically:

$$\Delta = \pm g^2. \tag{A2.13}$$

In the treatment of this paper we used the value $\Delta = g^2$ since in the case of $\Delta = -g^2$—according to (A2.12a)—from the relation $p'^2 = \kappa^2 - g^2$ we find imaginary values of p'_0 if $g^2 > \kappa^2$ (long-wavelength photons) which is physically inadmissible. The concrete form of w is derived in the text (equation (3.14)).

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